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THE FROZEN-IN CONDITION FOR A DIRECTION FIELD, SMALL DENOMINATORS AND CHAOTIZATION OF STEADY FLOWS OF A VISCOUS FLUID†

V. V. KOZLOV

Moscow

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A classical theorem of Helmholtz states that vortex lines are frozen into a flow of barotropic ideal fluid in a potential force field. This result leads to the following general problem: it is required to find conditions under which a given dynamical system admits of a direction field frozen into its phase flow. By the rectification theorem for trajectories, a whole family of frozen direction fields always exists locally. It turns out that the problem of the existence of non-trivial frozen direction fields defined in the whole phase space is closely related to the well-known problem of small denominators. Results of a general nature are applied to Hamiltonian systems, and also to steady flows of a viscous fluid. © 1999 Elsevier Science Ltd. All rights reserved.

1. FROZEN-IN CONDITION FOR A DIRECTION FIELD

Let M be a smooth manifold, let $\mathbf{v}$ be a vector field on M generating a dynamical system

$$\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}), \quad \mathbf{x} \in M \tag{1.1}$$

and let g^t be the phase flow of the systems.

Let $\mathbf{a} \neq 0$ be another smooth vector field on M. Through every point $\mathbf{x} \in M$ there passes a unique integral curve of $\mathbf{a}$ (at each point $\mathbf{x}$ of this curve it is tangent to the vector $\mathbf{a}(\mathbf{x})$). We shall say that the family of integral curves is *frozen* into the flow of system (1.1) if it is mapped into itself under all transformations g^t.

A criterion for the integral curves of a field $\mathbf{a}$ to be frozen is the truth of the equality

$$[\mathbf{a}, \mathbf{v}] = \lambda \mathbf{a} \tag{1.2}$$

where [,] is the commutator of the vector fields and λ is some smooth function on M. In order to prove (1.2), we use the rectification theorem for integral curves of $\mathbf{a}$: in suitable local coordinates $x_1, \ldots, x_n$, the components of $\mathbf{a}$ have the form $1, 0, \ldots, 0$. Condition (1.2) is equivalent to the series of equalities

$$\partial v_1 / \partial x_1 = \lambda, \quad \partial v_2 / \partial x_1 = \ldots = \partial v_n / \partial x_1 = 0 \tag{1.3}$$

where v_i are the components of the field $\mathbf{v}$. Since the integral curves of the field $\mathbf{a}$ are described in these coordinates by the equations $x_k = \text{const}$, $k \geqslant 2$, while the components v_k, $k \geqslant 2$, are independent of x_1, it follows that this family of curves is mapped into itself under the transformations g^t, and conversely, if conditions (1.3) fail to hold, then certain of the components $v_2, \ldots, v_n$ of $\mathbf{v}$ take different values for different values of the coordinate x_1 and therefore the phase flow g^t will twist the coordinate curves $x_k = \text{const}$, $k \geqslant 2$.

Condition (1.3) for $n = 3$ was first established by Poincaré, Zhoravskii and Fridman (see [1, 2]) as an extension of Helmholtz's theorem according to which the vortex lines (integral curves of the field of the curl) are frozen into a flow of an ideal barotropic fluid in a potential force field. In the non-autonomous case, the integral curves of the field $\mathbf{a}(\mathbf{x}, t)$ are considered at fixed times t, and condition (1.2) is replaced by the more general condition

$$\partial \mathbf{a} / \partial t + [\mathbf{a}, \mathbf{v}] = \lambda \mathbf{a} \tag{1.4}$$

Obviously, Eq. (1.2) does not change its form if the field $\mathbf{a}$ is replaced by $\mu \mathbf{a}$, where μ is any smooth function of $\mathbf{x}$. Consequently, it does not depend on the magnitude of the vectors $\mathbf{a}(\mathbf{x})$. Thus, Eq. (1.2) may be considered as a condition for *a direction field to be frozen* into the phase flow of the field $\mathbf{v}$.

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A derivation of the frozen-in condition (1.4) for the integral curves of a vector field $\mathbf{a}(\mathbf{x}, t)$ may also be found in the classical textbook [3].

If $\lambda = 0$, a will be a field of symmetries for system (1.1). In the general case, condition (1.2) may be given the following group-theoretic interpretation: the phase flow of the dynamical system (1.1) maps phase trajectories (but not solutions) of the dynamical system

$$d\mathbf{x}/d\alpha = \mathbf{a}(\mathbf{x}), \quad \alpha \in \mathbb{R}$$

into trajectories of the same system [4]. Note that, unlike the problem of fields of symmetries, the determination of frozen direction fields is a non-linear problem: apart from the field $\mathbf{a}$, the factor λ in (1.2) is also an unknown quantity.

The problem of whether frozen direction fields exist for a given system of differential equations was apparently first considered by Fridman [2, Section 10]. Fridman's method is actually based on expanding the solutions in series of powers of the time. Hence the results established in [2] are local in nature (both with respect to the space variables $\mathbf{x}$ and with respect to the time t). Moreover, for autonomous systems of type (1.1), Fridman's local series yield field $\mathbf{a}$ that depend explicitly on time. At the same time, the rectification theorem for trajectories of system (1.1) may be used to obtain families of non-trivial vector fields $\mathbf{a}$ that do not depend on t and satisfy (1.2). Moreover, these local results have very little value for dynamics. From the contemporary point of view, which goes back to Poincaré [5], it is useful to consider objects (such as first integrals, fields of symmetries, etc.) which are uniquely defined in the whole phase space M or in a part of it where the trajectories of system (1.1) have the recurrence property.

2. SMALL DENOMINATORS

Consider the following system of differential equations

$$\dot{x} = u_0 + \varepsilon u_1 + \ldots, \quad \dot{y} = v_0 + \varepsilon v_1 + \ldots, \quad \dot{z} = \varepsilon w_1 + \ldots \tag{2.1}$$

where the right-hand sides are series in powers of ε whose coefficients are analytic functions of x, y, z, 2π-periodic in x and y. It is assumed that u_0 and v_0 are functions of z only. It may be assumed that the phase space M of system (2.1) is a direct product $\Delta \times \mathbb{T}^2$, where Δ is the interval in which the variable z varies and $\mathbb{T}^2 = \{x, y \bmod 2\pi\}$ is a two-dimensional torus.

If $\varepsilon = 0$, we have a completely integrable system. The z coordinate is a first integral, whose level surfaces are two-dimensional surfaces carrying conditionally-periodic trajectories with two frequencies u_0 and v_0.

Systems of type (2.1) are one of the key objects of non-linear oscillation theory [6]. In particular, Hamiltonian systems with two degrees of freedom which are perturbations of integrable systems are reducible to this form (after iso-energetic reduction). Poincaré termed the study of systems of this type the fundamental problem of dynamics [5, Section 13].

Let us consider the problem of whether a vector field

$$\mathbf{a} = \mathbf{a}_0 + \varepsilon \mathbf{a}_1 + \ldots \tag{2.2}$$

exists for system (2.1) satisfying condition (1.2), and moreover such that the vector fields $\mathbf{a}_0, \mathbf{a}_1, \ldots$ are single-valued and analytic $\Delta \times \mathbb{T}^2$. Of course, the function λ must also be sought as a power series $\lambda_0 + \varepsilon\lambda_1 + \ldots$ with single-valued analytic coefficients.

Let us assume that $v_0 \neq 0$. We will call the unperturbed *non-degenerate* if the frequency ratio u_0/v_0 is a non-constant function on Δ. An equivalent condition is: $u'_0 v_0 - u_0 v'_0 \not\equiv 0$ (where the prime denotes differentiation with respect to z).

We expand the function w_1 in a Fourier series

$$w_1 = \sum_{-\infty}^{\infty} W_{mn}(z) \exp[i(mx + ny)]$$

We now introduce the Poincaré set $\mathbb{P}$, defined as the set of all points $z \in \Delta$ such that

$$1)\ m u_0(z) + n v_0(z) = 0, \quad m^2 + n^2 \neq 0$$

$$2)\ W_{mn}(z) \neq 0$$

The points of the Poincaré set correspond to resonant tori of the unperturbed problem which are destroyed when a perturbation is added. In the typical situation, the set $\mathbb{P}$ is everywhere dense in the interval Δ, a fact closely related to the problem of small denominators, which plays an important role in the investigation of system (2.1) [7].

A frozen direction field is said to be *trivial* if $\mathbf{a} = \mu\mathbf{v}$. In that case the phase trajectories of system (1.1) turn out to be frozen. In the problem under consideration, the fields $\mathbf{v}$ and $\mathbf{a}$ depend on the parameter ε; we will assume that the direction field satisfies the non-triviality condition at $\varepsilon = 0$: $\mathbf{a}_0 \neq \mathbf{v}_0$. In accordance with Section 1, we will also assume that $\mathbf{a}_0 \neq 0$. Otherwise, some of the integral curves of the field $\mathbf{a}_0$ will lose the regularity property and degenerate into points.

The main result of this paper is the following.

Theorem 1. Suppose the unperturbed system is non-degenerate and that the Poincaré set has at least one limit point in the interior of Δ. Then Eqs (2.1) do not admit of non-trivial frozen direction fields which are analytic in ε.

It has been proved [8] that, under the assumptions of Theorem 1, system (2.1) does not admit of non-constant integrals and non-trivial fields of symmetries which are series in powers of ε with analytic coefficients. If one requires in addition that $W_{00}(z) \not\equiv 0$, then system (2.1) does not admit of non-trivial linear integral variants $\oint\varphi_\varepsilon$, where the 1-form φ_ε is analytic in ε and $d\varphi_\varepsilon \neq 0$ [9]. The coefficient W_{00} is obviously equal to the mean value of the function w_1 over the two-dimensional torus $\mathbb{T}^2$. It has been proved [10] that the condition $W_{00} \not\equiv 0$ is essential: it does not hold for Hamiltonian systems, and such systems admit of a non-trivial Poincaré–Cartan integral variant. One possible interpretation of the non-existence of integral invariants of system (2.1) is that there is no analogue of Thomson's theorem on the conservation of circulation in an ideal liquid around a closed contour frozen into the flow.

We now prove Theorem 1. According to (2.1), the field $\mathbf{v}$ may be expanded in a series $\mathbf{v}_0 + \varepsilon\mathbf{v}_1 + \dots$, where the components of the field $\mathbf{v}_0$ are μ_0, υ_0, 0. Let $\mathbf{a}$ have the form (2.2); denote the components of $\mathbf{a}_0$ by a_0, b_0 and c_0. Setting $\varepsilon = 0$ in (1.2), we obtain three equations

$$\begin{aligned} &c_0\partial u_0/\partial z - u_0\partial a_0/\partial x - v_0\partial a_0/\partial y = \lambda_0 a_0 \\ &c_0\partial v_0/\partial z - u_0\partial b_0/\partial x - v_0\partial b_0/\partial y = \lambda_0 b_0 \\ &-u_0\partial c_0/\partial x - v_0\partial c_0/\partial y = \lambda_0 c_0 \end{aligned} \tag{2.3}$$

Fix a value of $z = z_0 \in \Delta$. The last equation of system (2.3) may be rewritten in the form

$$\dot{c}_0 = -\lambda_0 c_0 \tag{2.4}$$

where the dot denotes the total derivative of the function c_0: $\mathbb{T}^2 \to \mathbb{R}$ along trajectories of the system on the torus

$$\dot{x} = u_0(z_0), \quad \dot{y} = v_0(z_0) \tag{2.5}$$

Let us assume that the torus $z = z_0$ is non-resonant. If the function c_0 vanishes at some point $(x_0, y_0) \in \mathbb{T}^2$, then (since Eq. (2.4) is linear) it vanishes on the whole trajectory of system (2.5) passing through the point $x = x_0$, $y = y_0$. By assumption, at $z = z_0$ the frequency ratio u_0/υ_0 is irrational. Consequently, all the trajectories of system (2.5) are everywhere dense on the torus and by continuity $c_0 \equiv 0$.

Let us assume now that $c_0 \neq 0$ at $z = z_0$. Setting $\nu = a_0/c_0$, we deduce from the first and third equations of system (2.3) that

$$u_0\partial\nu/\partial x + v_0\partial\nu/\partial y = \partial u_0/\partial z$$

or, which is the same thing, that $\dot{\nu} = \partial u_0/\partial z$. Since the right-hand side of this equality is independent of x and y, it follows that

$$(\nu(t) - \nu(0))/t = \partial u_0/\partial z$$

Letting $t \to \infty$ and using the fact that ν is bounded, we see that $\partial u_0/\partial z = 0$ at $z = z_0$. An analogous argument holds for the derivative $\partial v_0/\partial z$ also.

Thus, the relation $(u_0'\upsilon_0 - u_0\upsilon_0')c_0 = 0$ holds on a non-resonant subset of Δ. By continuity, this relation holds everywhere on $\Delta \times \mathbb{T}^2$. As there are no divisors of zero in the ring of analytic functions, one factor must vanish identically. Since we have assumed non-degeneracy, it follows that $c_0 \equiv 0$.

When $c_0 \equiv 0$ the first two equations of (2.3) have the same form as the third equation. Consequently, the functions a_0 and b_0 either vanish identically on non-resonant tori or, on the contrary, have no zeros at all there. Suppose, say, that $a_0 \neq 0$. Then the quotient $\varkappa = b_0/a_0$ satisfies the equation

$$u_0 \partial\varkappa/\partial x + v_0 \partial\varkappa/\partial y = 0$$

Consequently, b_0/a_0 = const on non-resonant tori.

By assumption, $a_0^2 + b_0^2 \neq 0$. We may therefore set $a_0 = r\xi$, $b_0 = r\eta$, where

$$r = (a_0^2 + b_0^2)^{1/2}, \quad \xi = a_0/r, \quad \eta = b_0/r$$

with r, ξ and η analytic functions on $\Delta \times \mathbb{T}^2$. Since ξ and η in fact depend on the quotient b_0/a_0, they are constant on non-resonant tori. Since the non-resonant tori are everywhere dense, it follows that ξ and η are analytic functions of z only.

Let a_1, b_1 and c_1 be the components of the vector field $\mathbf{a}_1$. Equating the coefficients of ε in (1.2) to zero, we obtain three equations; one of these is the equation for the z coordinate

$$a_0 \partial w_1/\partial x + b_0 \partial w_s/\partial y - u_0 \partial c_1/\partial x - v_0 \partial c_1 \tag{2.6}$$

Since $r \neq 0$, we may assume that $c_1 = \sigma r$, where σ is some analytic function on $\Delta \times \mathbb{T}^2$. It follows from (2.3) that r satisfies the equation

$$-u_0 \partial r/\partial x - v_0 \partial r/\partial y = \lambda_0 r \tag{2.7}$$

Setting $a_0 = r\xi$, $b_0 = r\eta$ and using (2.7), we can reduce Eq. (2.6) to the following form

$$\xi \partial w_1/\partial x + \eta \partial w_1/\partial y - u_0 \partial\sigma/\partial x - v_0 \partial\sigma/\partial y = 0$$

This linear equation is solved by Fourier's method. Equating the coefficients of like harmonics, we obtain an infinite chain of simple algebraic equations

$$(m\xi + n\eta)W_{mn} = (mu_0 + nv_0)\Sigma_{mn} \tag{2.8}$$

where $\Sigma_{mn}(z)$ are the Fourier coefficients of σ.

Now let $z \in \mathbb{P}$. Then $mu_0 + nv_0 = 0$, and it follows from (2.8) that $m\xi + n\eta = 0$. Since $m^2 + n^2 \neq 0$, the determinant of this linear system, $f = u_0\eta - v_0\xi$, vanishes. The function f is analytic on Δ and its zeros have a limit point in the interior of Δ. Consequently, $f \equiv 0$. Thus, when $\varepsilon = 0$ the vectors $\mathbf{v}$ and $\mathbf{a}$ are linearly dependent at all points of the phase space. This completes the proof of the theorem.

Let the set M be compact and assume that system (1.1) is ergodic. Then trajectories exist that fill out M everywhere densely; in particular, such systems do not admit of non-constant first integrals. However, the ergodic property does not contradict the existence of non-trivial fields of symmetries.

Here is a simple example: let M be the n-dimensional torus $\{x_i \bmod 2\pi\}$ and let the system be given by equations

$$\dot{x}_1 = \omega_1, \ldots, \dot{x}_n = \omega_n \tag{2.9}$$

with constant incommensurable frequencies ω. It is clear that any vector field with constant components is a field of symmetries. The ergodic system (2.9) is degenerate in a certain sense: its entropy is zero. An example of the opposite property is provided by Anosov systems [11] with unstable behavior of the phase trajectories. In particular, all periodic trajectories are hyperbolic and the set of all such trajectories is everywhere dense in the whole phase space.

It has been shown [8] that Anosov systems do not admit of non-trivial fields of symmetries. They may, however, have non-trivial frozen direction fields.

Here is a simple example (cf. [12, Section 14]). Consider the three-dimensional manifold M obtained from the direct product of the torus $\mathbb{T}^2 = \{x_1, x_2 \bmod 2\pi\}$ and the interval $0 \leqslant x_3 \leqslant 1$ by gluing the end tori together according to the following rule: a point $(x_1, x_2, 1)$ is identified with the point $(x_1', x_2', 0)$, where

$$x_1' = 2x_1 + x_2, \quad x_2' = x_1 + x_2 \pmod{2\pi} \tag{2.10}$$

Consider the vector field $\mathbf{v}$ on $\mathbb{T}^2 \times \{0, 1\}$ with components 0, 0, 1. After gluing, this field is converted into a smooth field on M which defines an Anosov system. Define $\mathbf{a} = (a_1, a_2, 0)$, where (a_1, a_2) is an eigenvector of the linear mapping (2.10) (there are in fact two linearly independent eigenvectors). Clearly, the field $\mathbf{a}$ generates a non-trivial direction field which is frozen in to the flow of $\mathbf{v}$. It should be noted that the integral curves of $\mathbf{a}$ fill out the two-dimensional tori x_3 = const everywhere densely.

3. SOME APPLICATIONS

Fridman ([2, Section 9]) found conditions under which the vortex lines of a field **v** are frozen into the flow g^t, as well as conditions for the conservation of the circulation of **v** around any closed contour. In that case, M is an ordinary Euclidean three-dimensional space. Fridman's conditions are local in nature. He presented examples of fields for which the circulation remains unchanged but the vortex lines are not frozen [2].

We will now present a contrasting example, relating to the dynamics of a homogeneous incompressible fluid in a potential field of external forces, taking into account viscous friction in Rayleigh's form. The equations of motion are

$$\partial \mathbf{v}/\partial t + (\operatorname{rot}\mathbf{v})\times\mathbf{v} = -\partial f/\partial \mathbf{x} - k\mathbf{v} \tag{3.1}$$

where f is the Bernoulli trinomial and k is the coefficient of viscous friction. Applying the curl operator to both sides of (3.1) and using formulae of vector analysis, as well as the incompressibility of the fluid (div **v** = 0), we obtain the equality

$$\partial \mathbf{a}/\partial t + [\mathbf{a}, \mathbf{v}] = -k\mathbf{a}, \quad \mathbf{a} = \operatorname{rot}\mathbf{v}$$

Consequently, by (1.4), the field lines of the curl of the velocity are frozen into the flow.

Now let γ be a closed contour, we define

$$I(t) = \int_{g^t\gamma} (\mathbf{v}, d\mathbf{x})$$

Equations (3.1) imply a relation according to which the circulation varies exponentially: $I(t) = I(0)\exp(-kt)$.

Fridman's problem may be generalized, comparing the conditions for existence in the large of frozen direction fields and integral invariants of dynamical systems on three-dimensional manifolds. To that end, consider the Hamiltonian system

$$\dot{x} = 1, \quad \dot{y} = \partial H/\partial z, \quad \dot{z} = -\partial H/\partial y; \quad H = H_0(z) + \varepsilon H_1(x, y, z) + \ldots \tag{3.2}$$

Here $y \bmod 2\pi$, z are action-angle variables for the unperturbed system, and the function H is assumed to be 2π-periodic in the "time" $x = t$. Systems of type (3.2) are obtained from autonomous systems with two degrees of freedom after reducing the order in Whittaker's sense.

For system (3.2)

$$u_0 = 1, \quad v_0 = \partial H_0/\partial z, \quad w_1 = -\partial H_1/\partial y$$

Consequently, the condition for the unperturbed system to be non-degenerate is equivalent to the inequality $d^2H_0/dz^2 \not\equiv 0$, and the Poincaré set $\mathbb{P}$ is

$$\{z \in \Delta : dH_0/dz = -n/m, \; H_{mn} \neq 0\}$$

where H_{mn} are the Fourier coefficients of the perturbing function H_1. In a typical situation, $\mathbb{P}$ is everywhere dense in Δ. Consequently, by Theorem 1, Eq. (3.2) has no non-trivial frozen direction fields. But it always has a Poincaré–Cartan invariant

$$\oint z\,dy - H\,dx$$

Let us generalize this situation slightly. Let M be a three-dimensional manifold and suppose that system (1.1) admits of a non-trivial integral invariant on M

$$\oint \varphi \tag{3.3}$$

where φ is a 1-form, $d\varphi \neq 0$. The invariance condition (3.3) becomes

$$L_\mathbf{v}\varphi = dg \tag{3.4}$$

where L_v is the Lie derivative and g is a scalar function on M. By the homotopy formula

$$L_\mathbf{v} = di_\mathbf{v} + i_\mathbf{v}d$$

($i_\mathbf{v}$ is the inner product of the field **v** and a differential form). Consequently, the relationship (3.4) becomes

$$l_{\mathbf{v}}\Phi = dh; \quad \Phi = d\varphi, \quad h = g - \varphi(\mathbf{v})$$

Since $\Phi \neq 0$ and the set M is three-dimensional, it follows that at every point there is a non-zero tangent vector $\mathbf{a}(x)$ such that $i_{\mathbf{a}}\Phi = 0$. This vector is defined uniquely apart from a constant factor. It can be shown that the integral curves of $\mathbf{a}$ are frozen into the flow of system (1.1). Of course, it may turn out that h = const. Then the field $\mathbf{a}$ is collinear with the field $\mathbf{v}$ and the frozen direction field will be trivial. This is precisely the situation in the case of Hamiltonian systems. However, if h is not an integral of the field $\mathbf{a}$, this field will generate a non-trivial frozen direction field.

4. CHAOTIZATION OF STEADY FLOWS OF A VISCOUS FLUID

Consider the steady flow of a viscous incompressible fluid in the Stokes approximation [13]. To simplify matters, let us assume that there are no external forces. The equations of motion take the form

$$\partial p / \partial \mathbf{x} = \mu \Delta \mathbf{v}, \quad \operatorname{div} \mathbf{v} = 0 \tag{4.1}$$

where p is the pressure and μ is the coefficient of dynamic viscosity, which we shall assume is equal to one (for example, by making the substitution $p \to p/\mu$).

We will seek solutions of system (4.1) in the form

$$u = u_0 + \varepsilon u_1, \; v = v_0 + \varepsilon v_1, \quad w = \varepsilon w_1, \quad p = p_0 + \varepsilon p_1 \tag{4.2}$$

where ε is a parameter, and the functions u_0, v_0 and p_0 depend only on z. Solutions of this type with $\varepsilon = 0$ are important in meteorology [2]. It is assumed that the functions u_1, v_1, w_1 and p_1 are 2π-periodic in x and y.

Substituting (4.2) into (4.1), we obtain the relations

$$u_0 = \alpha z + \xi, \quad v_0 = \beta z + \eta, \quad p_0 = \text{const}$$

The coefficients α, β, ξ, η are constant; we will assume that

$$\alpha\eta - \beta\xi \neq 0 \tag{4.3}$$

Let U_{mn}, V_{mn}, W_{mn}, P_{mn} be the Fourier coefficients of the functions u_1, v_1, w_1, p_1. They depend on z and are found from the following linear system

$$\begin{aligned} U''_{mn} &= (m^2 + n^2)U_{mn} + imP_{mn}, \quad V''_{mn} = (m^2 + n^2)V_{mn} + inP_{mn} \\ P'_{mn} &= -(m^2 + n^2)W_{mn} - i(mU'_{mn} + nV'_{mn}) \\ W'_{mn} &= -i(mU_{mn} + nV_{mn}) \end{aligned} \tag{4.4}$$

(the prime denotes differentiation with respect to z). Equations (4.4) may be regarded as a linear system of ordinary second-order differential equations, of second order in U and V and first order in P and W. They have solutions in every interval Δ on the $\{z\}$ axis which take given values at a fixed point of Δ (as do the derivatives U', V'). Since the linear system (4.4) is closed at fixed values of m, n, the construction of convergent Fourier series presents no difficulties.

Thus, the field of velocities (4.2) has the form (2.1). Resonant tori $mu_0 + nv_0 = 0$ correspond to points

$$z_{mn} = -(m\xi + n\eta)/(m\alpha + n\beta)$$

By (4.3), they are everywhere dense on the $\{z\}$ axis. For typical flows, the value of W_{mn} at the points z_{mn} do not vanish. In the general case, therefore the Poincaré set is dense on the real line $\mathbb{R} = \{z\}$. Condition (4.3) is also the condition for the unperturbed system to be non-degenerate. Thus, by Theorem 1, a typical steady flow (4.2) has no non-trivial frozen direction fields.

Since the fluid is incompressible, the density is a first integral. By a well-known result [8], a typical field (4.2) does not admit of non-constant first integrals. In this case, therefore, a viscous fluid is necessarily homogeneous.

Remark. Solutions of the full Navier–Stokes equations may also be found as formal series in powers of ε, but one then has the non-trivial task of proving that they are convergent [9].

5. CONCLUDING REMARKS

Theorem 1 asserts that there are no frozen direction fields that are analytic with respect to ε. The assumption as to analyticity with respect to ε may probably be dropped, but this still awaits proof. An even simpler problem remains to be solved: it is required to prove that, under the assumptions of Theorem 1, for small values of $\varepsilon \neq 0$, differential equations (2.1) do not admit of non-constant analytic integrals. Problems of this kind are very difficult. Suffice it to recall the result, arising from KAM-theory, that for small $\varepsilon \neq 0$ Hamiltonian systems of type (2.1) always have a non-constant continuous first integral [14]. On the other hand, there are examples of Hamiltonian systems that admit of a C^k integral but have no C^{k+1} integrals defined throughout the phase space [7].

In many cases, splitting of separatrices [7, 14] may be used to prove that there are no analytic integrals for fixed small values of $\varepsilon \neq 0$. However, splitting of separatrices does not preclude the existence of non-trivial frozen direction fields. Indeed, in the example of Section 2, a system on M^3 has infinitely many hyperbolic periodic trajectories whose separatrices intersect transversely.

Examples have been given [9] of steady flows of viscous fluid with separatrix crossing. The chaotic structure of some flows of viscous fluid has been investigated [15].

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